University of Caloocan City

*College of Liberal Arts & Sciences*

**Mathematics Department**

**Integral Calculus: Learning Module No. 7**

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| --- | --- |
| **Topic** | **METHODS OF INTEGRATION** |
| **Sub-Topic** | **Trigonometric Substitution** |
| **Duration** | 3 hours |
| **Introduction** | There are certain types of integrals involving algebraic expressions which can be transformed into a problem of evaluating trigonometric integrals. The transformation will involve appropriate trigonometric substitutions for the original variable of  integration. |
| **Theories/Concepts/Formulas**  Note: This module may contain copyrighted material. The use of which has not been specifically authorized by the copyright owner. This module is for educational purpose only for online instruction and is not used to generate profit. Thus this constitutes a “Fair Use” of the copyrighted material as provided by virtue of Republic Act No. 8293 otherwise known as Intellectual Property Code of the Philippines. | 1. **Trigonometric Substitution**   The aforesaid trigonometric substitutions which will lead to integrable forms are listed as follows:   * 1. If the integrand contains 𝒂𝟐 − 𝒖𝟐   use the substitution 𝒖 = 𝒂 𝒔𝒊𝒏.   * 1. If the integrand contains 𝒂𝟐 + 𝒖𝟐   use the substitution 𝒖 = 𝒂 𝒕𝒂𝒏.   * 1. If the integrand contains 𝒖𝟐 − 𝒂𝟐   use the substitution 𝒖 = 𝒂 𝒔𝒆𝒄 |
| **YouTube Link/s** | <https://www.youtube.com/watch?v=cmZWy3GmQw4> <https://www.youtube.com/watch?v=DWWa7dSZmR8> <https://www.youtube.com/watch?v=Fx10QvjpZrc> <https://www.youtube.com/watch?v=GVL4lHX6DgM>  <https://www.youtube.com/watch?v=3lC5AuCFK4c> |
| **Sample Problems** | 𝑑𝑥  Ex.1. Evaluate ∫  √𝑎2−𝑥2  *a*2 = *a* 2 u2 = x2 |

*a* = *a* u = x

use: u = *a* sin

x = *a* sin dx= *a* cos d

*a*

𝑠𝑖𝑛 =

𝑥

𝑎

x



√𝑎2 − 𝑥2

√𝑎2 − 𝑥2

√𝑎2 − 𝑥2 = 𝑎 𝑐𝑜𝑠

𝑑𝑥

∫ √𝑎2−𝑥2=∫

𝑎 𝑐𝑜𝑠  𝑑

𝑎 𝑐𝑜𝑠 

but and

=∫ 𝑑

=  + 𝐶

𝑠𝑖𝑛 = 𝑥

𝑎

 = 𝐴𝑟𝑐𝑠𝑖𝑛 𝑥

𝑎

Thus,

Ex.2. Evaluate ∫

(𝑥+3)𝑑𝑥

√16−𝑥2



∫

𝒅𝒙

√𝒂𝟐−𝒙𝟐

= 𝑨𝒓𝒄𝒔𝒊𝒏 + 𝑪

𝒙

𝒂

𝑐𝑜𝑠  =

𝑎

*a* 2 = 16 u2 = x2

*a* = 4 u = x

use: u = *a* sin



∫ = 𝟑𝑨𝒓𝒄𝒔𝒊𝒏 − √𝟏𝟔 − 𝒙𝟐 + 𝑪

(𝒙 + 𝟑)𝒅𝒙

𝒙

√𝟏𝟔 − 𝒙𝟐

𝟒

x = 4 sin dx= 4 cos d

𝑠𝑖𝑛 = 𝑥 4

x

4 

√16 − 𝑥2

𝑐𝑜𝑠  =

√16 − 𝑥2

4

√16 − 𝑥2 = 4 𝑐𝑜𝑠

(𝑥+3)𝑑𝑥

∫ √16−𝑥2 =∫

(4𝑠𝑖𝑛 +3)(4𝑐𝑜𝑠 𝑑) 4 𝑐𝑜𝑠 

but

and

=∫(4𝑠𝑖𝑛 + 3)𝑑

= 4 ∫ 𝑠𝑖𝑛 𝑑 + 3 ∫ 𝑑

= 4(−𝑐𝑜𝑠 ) + 3 + 𝐶

−𝑐𝑜𝑠  = − √16−𝑥2

4

 = 𝐴𝑟𝑐𝑠𝑖𝑛 𝑥

4

√16 − 𝑥2 𝑥

= 4 (−

Thus,

) + 3(𝐴𝑟𝑐𝑠𝑖𝑛

4

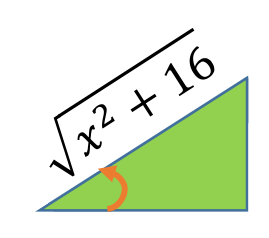
) + 𝐶

4

Ex.3. Evaluate ∫

𝑑𝑥

3



(𝑥2+16)2

𝑑𝑥

=∫ 3

(√𝑥2+16)

*a* 2 = 16 u2 = x2

*a* = 4 u = x

use: u = *a* tan

x = 4 tan dx= 4 sec2 d

𝑡𝑎𝑛 = 𝑥

4

 x

4

𝑠𝑒𝑐  =

4

√𝑥2 + 16

𝑑𝑥

∫

√𝑥2 + 16 = 4 𝑠𝑒𝑐

4𝑠𝑒𝑐2 𝑑

3 = ∫

(√𝑥2+16)

(4𝑠𝑒𝑐)3

4𝑠𝑒𝑐2 𝑑

=∫ 64𝑠𝑒𝑐3

4 𝑠𝑒𝑐2

= ∫(

64

1

𝑠𝑒𝑐3

𝑑

𝑑

= 16 ∫ 𝑠𝑒𝑐

from reciprocal identity

∫

𝒅𝒙

𝒙

𝟑

(𝒙𝟐+𝟏𝟔)𝟐

**=**

𝟏

𝟏𝟔(𝒙𝟐+𝟏𝟔)𝟐

+ 𝑪

1

𝑠𝑒𝑐

= 𝑐𝑜𝑠

but

= 1 ∫ 𝑐𝑜𝑠 𝑑

16

= 1 (𝑠𝑖𝑛) + 𝐶

16

𝑠𝑖𝑛 = 𝑥

√𝑥2+16

1 𝑥

= ( ) + 𝐶

16 √𝑥2+16

Thus,

𝑑𝑥

Ex.4. Evaluate ∫ 𝑥2+8𝑥+17

𝑑𝑥

=∫ 𝑥2+8𝑥+16+1

𝑑𝑥

=∫ (𝑥2+8𝑥+16)+1

𝑑𝑥

=∫ (𝑥+4)2+1

*a*2 = 1 u2 = (x+4)2

*a* = 1 u = x+4

use: u = *a* tan

x+4 = tan dx= sec2 d (x+4)2 = tan2

𝑡𝑎𝑛 = 𝑥 + 4



∫

𝒅𝒙

𝒙𝟐 + 𝟖𝒙 + 𝟏𝟕

= 𝑨𝒓𝒄𝒕𝒂𝒏 (𝒙 + 𝟒) + 𝑪

𝑑𝑥



1

𝑠𝑒𝑐2 𝑑

X+4

∫ 𝑥2+8𝑥+17 =∫ 𝑡𝑎𝑛2 +1

but from trigonometric identities

tan2 + 1 = sec2

𝑠𝑒𝑐2 𝑑

=∫ 𝑠𝑒𝑐2

from and

=∫ 𝑑

=  + 𝐶

𝑥 + 4 = 𝑡𝑎𝑛

 = 𝐴𝑟𝑐𝑡𝑎𝑛 (𝑥 + 4)

Thus,

Ex.5. Evaluate ∫

𝑥2𝑑𝑥

√4𝑥2−9

*a* 2 = 9 u2 = 4x2 = (2x)2

*a* = 3 u = 2x use: u = *a* sec

2x = 3sec, 2dx= 3sec tan d



3 3

𝑥 = 𝑠𝑒𝑐, 𝑑𝑥 = 𝑠𝑒𝑐 𝑡𝑎𝑛 𝑑

2 2

2 9 2

𝑥 = 𝑠𝑒𝑐 

4

𝑠𝑒𝑐 = 2𝑥

3

2x √4𝑥2 − 9

3

3

𝑡𝑎𝑛  = √4𝑥2−9 , √4𝑥2 − 9 = 3𝑡𝑎𝑛

3

9 2 3

𝑥2𝑑𝑥

( 𝑠𝑒𝑐 )(

𝑒𝑐 𝑡𝑎𝑛 𝑑)

∫ =∫ 4

4𝑥 −9

√ 2

9

𝑠𝑒𝑐3 𝑑

𝑠

2

3 𝑡𝑎𝑛 

= ∫

8

9

𝑠𝑒𝑐 . 𝑠𝑒𝑐2 𝑑

= ∫

8

u = sec dv = sec2 d

du = sec tan v = tan

∫ 𝑠𝑒𝑐3 𝑑 = 𝑠𝑒𝑐 𝑡𝑎𝑛

− ∫(tan  )(sec  tan  𝑑)

∫ 𝑠𝑒𝑐3 𝑑 = 𝑠𝑒𝑐 𝑡𝑎𝑛

− ∫ 𝑡𝑎𝑛2 . 𝑠𝑒𝑐 𝑑

but from trigonometric identities

tan2 = sec2 - 1

∫ 𝑠𝑒𝑐3 𝑑 = 𝑠𝑒𝑐 𝑡𝑎𝑛 − ∫(𝑠𝑒𝑐2 − 1) 𝑠𝑒𝑐 𝑑

∫ 𝑠𝑒𝑐3 𝑑 = 𝑠𝑒𝑐 𝑡𝑎𝑛 − ∫ 𝑠𝑒𝑐3 𝑑

=

𝒙√𝟒𝒙𝟐 − 𝟗

𝟖

+ 𝒍𝒏 (

𝟐𝒙 + √𝟒𝒙𝟐 − 𝟗

𝟑

𝟗

𝟏𝟔

) + 𝑪

+ ∫ 𝑠𝑒𝑐 𝑑

2 ∫ 𝑠𝑒𝑐3 𝑑 = 𝑠𝑒𝑐 𝑡𝑎𝑛 + ∫ 𝑠𝑒𝑐 𝑑

2 ∫ 𝑠𝑒𝑐3 𝑑 = 𝑠𝑒𝑐 𝑡𝑎𝑛 + ln(𝑠𝑒𝑐 + tan ) + 𝐶

∫ 𝑠𝑒𝑐3 𝑑 1  𝑡𝑎𝑛 + ln(𝑠𝑒𝑐 + tan )] + 𝐶 2

= [𝑠𝑒𝑐

= 9 . 1 [𝑠𝑒𝑐 𝑡𝑎𝑛 + ln(𝑠𝑒𝑐 + tan )] + 𝐶

8 2

9 2𝑥

= [(

16 3

√4𝑥2 − 9

) ( ) + ln ( 3

2𝑥

3

√4𝑥2 − 9

+

3

)] + 𝐶

9 2𝑥√4𝑥2 − 9 2𝑥 + √4𝑥2 − 9

= 16 [ 9

+ ln ( 3

)] + 𝐶

9 2𝑥√4𝑥2 − 9 9 2𝑥 + √4𝑥2 − 9

= 16 . 9 + 16 𝑙𝑛 ( 3 ) + 𝐶

𝑥√4𝑥2 − 9

= +

8

9

𝑙𝑛 (

16

2𝑥 + √4𝑥2 − 9

) + 𝐶

3

Thus, ∫

𝑥2𝑑𝑥

√4𝑥2−9